NAME: ____________________________

CLASS: 12MTX_____

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2020

YEAR 12

AP4

MATHEMATICS EXTENSION 1

Time allowed – 2 hours plus 10 minutes reading time

General Instructions

- Attempt all questions
- Write your name on the question paper
- Write using black pen
- Calculators approved by NESA may be used
- The NESA reference sheet has been provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless, badly arranged, or poorly written work

Total marks: 70

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1-10 on the answer sheet provided
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7 – 11)

- Attempt Questions 11-14
- Each question must be commenced in a new booklet clearly marked with your name, class and question number eg. Question 11, Question 12, Question 13, or Question 14
- Allow about 1 hour and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10.

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1. A PIN is to be chosen using four of the digits 0 through 9. Digits cannot be repeated.

How many possible PINs are there?

A. 256

B. 210

C. 10 000

D. 5040

2. An object's velocity is given by the vector $\binom{3}{-5}$.

What is the magnitude of the object's velocity?

A. -2

B. 34

C. $\sqrt{34}$

D. $\sqrt{8}$

3. The polynomial $P(x) = x^3 - 12x^2 + 21x + k$ has a double root.

What are the possible values of k?

A. $k = -1$ or $k = -7$

B. $k = 1$ or $k = 7$

C. $k = 10$ or $k = -98$

D. $k = -10$ or $k = 98$

4. What is the derivative of $\sin^{-1}3x$?

A. $-\frac{1}{\sqrt{1-9x^2}}$

B. $\frac{3}{\sqrt{1-9x^2}}$

C. $-\frac{1}{\sqrt{1-3x^2}}$

D. $-\frac{3}{\sqrt{1-9x^2}}$

5. Find the primitive function of $\frac{dy}{dx} = \sin x \cos^3 x$.

A. $y = \frac{1}{4}\cos^4 x + C$

B. $y = -\frac{1}{4}\cos^4 x + C$

C. $y = \frac{1}{8}\sin^2 x \cos^4 x + C$

D. $y = -\frac{1}{8}\sin^2 x \cos^4 x + C$

6. Write the expression $y = \sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$.

A. $2\sin\left(x + \frac{\pi}{4}\right)$

B. $\sin\left(x + \frac{\pi}{3}\right)$

C. $2\sin\left(x + \frac{\pi}{3}\right)$

D. $2\sin\left(x - \frac{\pi}{3}\right)$

7. What is the value of $\sin 2x$ given that $\sin x = \frac{2\sqrt{3}}{4}$ and x is obtuse?

A. $-\frac{\sqrt{3}}{4}$

B. $-\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{4}$

D. $\frac{\sqrt{3}}{2}$

8. The equation $y = e^{ax}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$

What are the possible values of a?

A. $a = -2$ or $a = 3$

B. $a = -1$ or $a = 6$

C. $a = 2$ or $a = -3$

D. $a = 1$ or $a = -6$

9. Given that $f(x) = e^x - 1$ and $y = f^{-1}(x)$, find an expression for $\frac{dy}{dx}$.

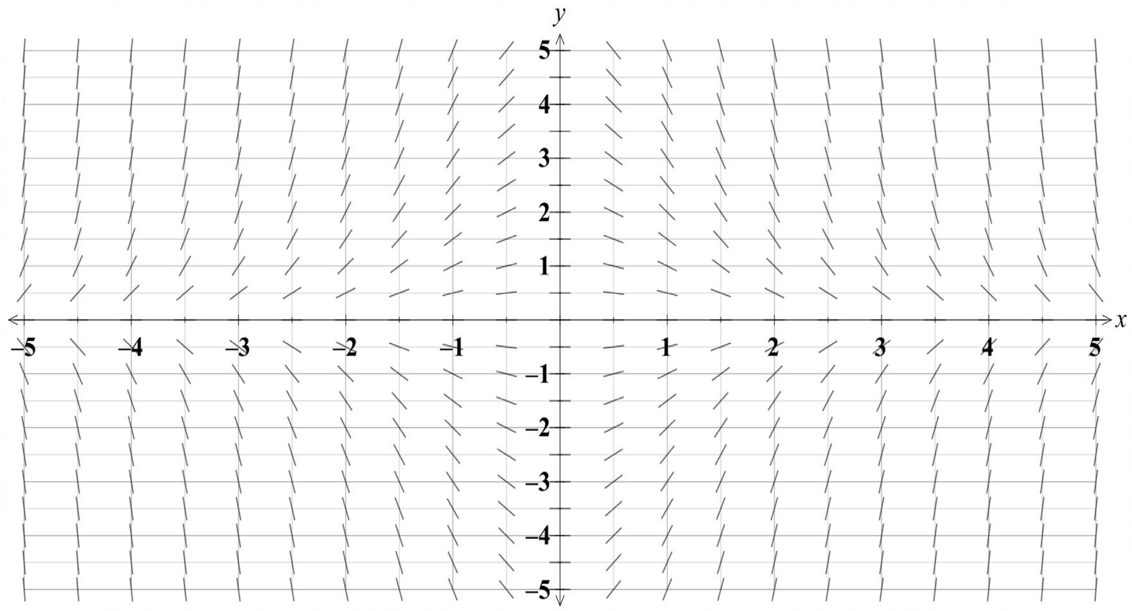
A. $\frac{1}{e^x-1}$

B. $\frac{1}{x+1}$

C. $\ln x$

D. $\ln(x+1)$

10. Which of the following differential equations could be represented by the slope field diagram below?



A. $y' = -xy$

B. $y' = xy$

C. $y' = -x^2y$

D. $y' = x^2y$

Section II

60 marks
Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 writing booklet.

(a) Write a cartesian equation to represent the following parametric equations. **2**

$x = 2\sin\theta$

$y = 2\cos\theta$

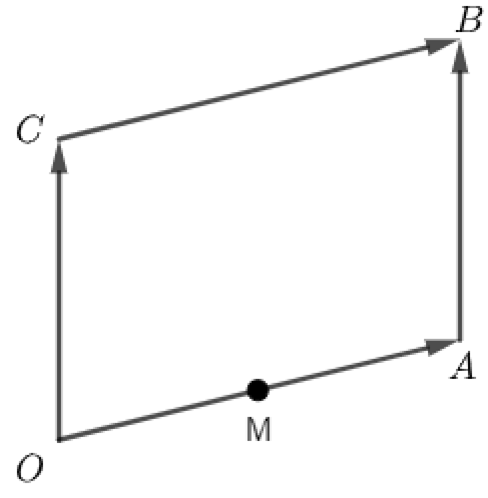
(b) Solve $\frac{1-x}{x-5} \geq 1$. **2**

(c) Consider the vectors $\underset{\sim}{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ onto $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

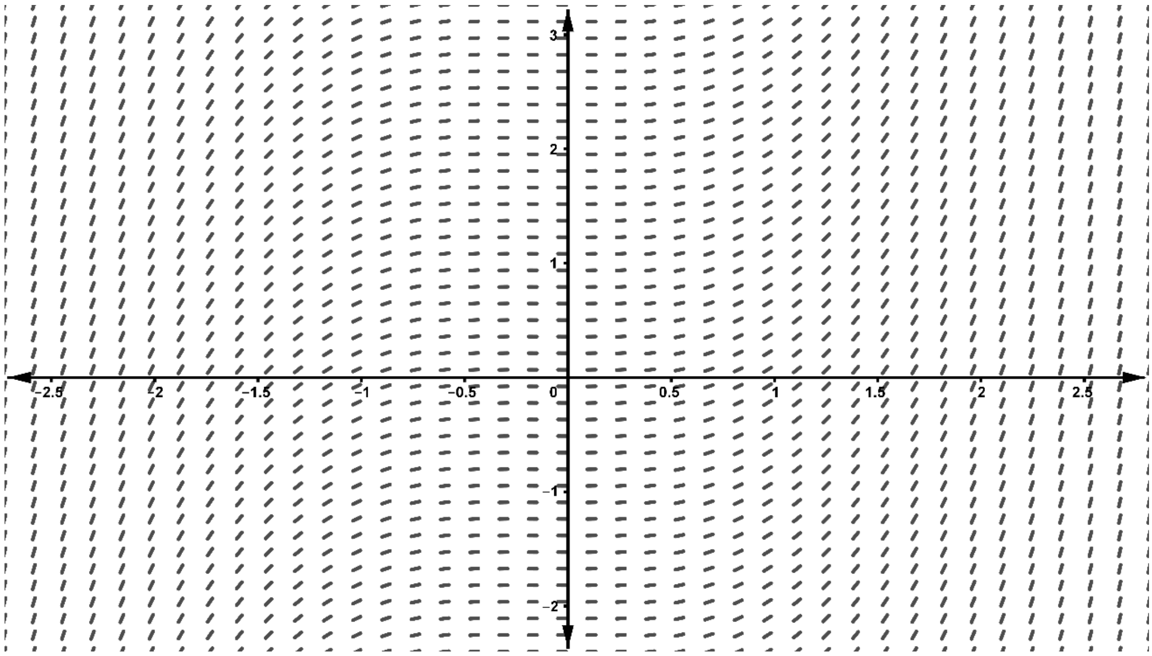
i) Using the dot product, show that these two vectors are not perpendicular to each other. **1**

ii) Find the vector projection of $\underset{\sim}{p}$ onto $\underset{\sim}{q}$. **2**

(d) In the diagram, *OABC* is a parallelogram. The vector $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. M is the midpoint of *OA*. Write an expression in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ to represent the vector $\overrightarrow{MB}$. **1**



(e) Sketch a possible solution curve passing through the origin for the following direction field. **1**



(f) Find $\int_0^{\ln 2} \frac{e^{2x}}{1+e^{4x}}\, dx$ by using the substitution $u = e^{2x}$. **3**

Give your answer correct to 4 decimal places.

(g) Use the substitution $t = \tan\frac{x}{2}$ to find in radians, correct to 2 decimal places, the solutions of the equation $\cos x - 3\sin x + 3 = 0$ for $0 \le x \le 2\pi$. **3**

End of Question 11

Question 12 (15 marks) Use the Question 12 writing booklet.

(a) i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$. **1**

ii) Hence, find **3**

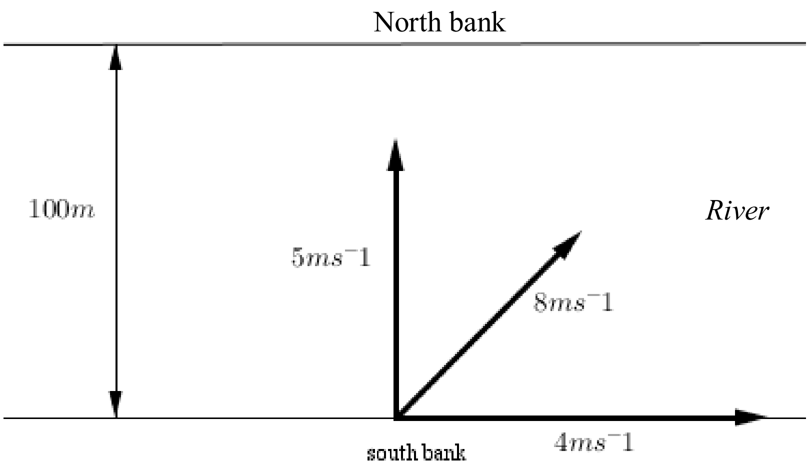
$$\int_0^{\frac{\pi}{2}} \sin 4x \cos 2x \, dx$$

(b) Find $\int_0^{\frac{5}{2}} \frac{1}{25+4x^2}\, dx$. Leave your answer in exact form. **2**

(c) The polynomial equation $x^3-5x^2+x+3=0$ has roots α, β and γ. **3**

Find $\alpha^2+\beta^2+\gamma^2$.

(d) Sienna intends to row her boat from the south bank of a river to meet with her friends on the north bank, as shown in the diagram below. The river is 100 metres wide. Sienna's rowing speed is 5 ms^{-1} when the water is still. The river is flowing east at a rate of 4 ms^{-1}. Sienna's boat is also being impacted by a wind blowing from the south-west, which pushes the boat at 8 ms^{-1}. She starts rowing across the river by steering the boat such that it is perpendicular to the south bank.



i) Show that the velocity of Sienna's boat can be expressed as the component vector: **2**

$(4+4\sqrt{2})\underset{\sim}{\mathbf{i}}+(5+4\sqrt{2})\underset{\sim}{\mathbf{j}}$

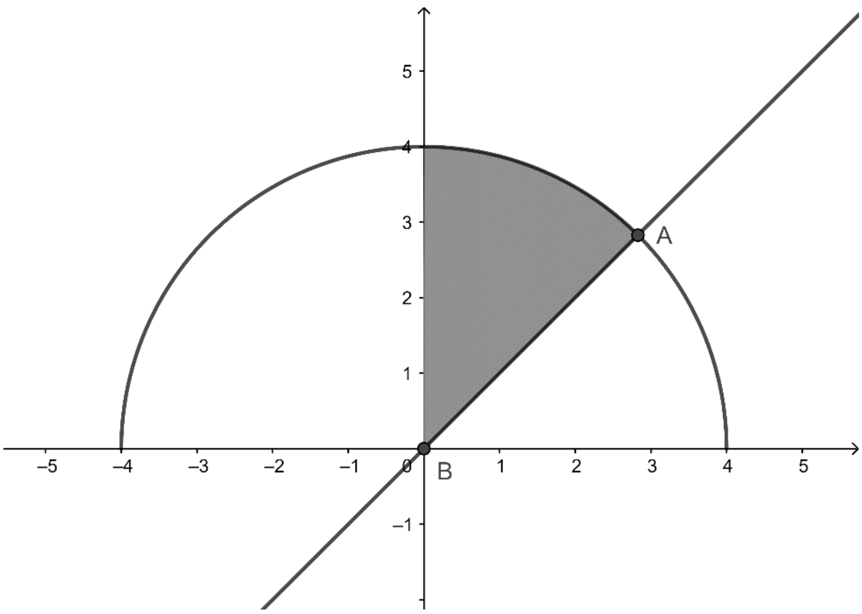
ii) Determine the distance rowed (in metres) and time taken (in seconds) by Sienna to travel from the starting point to her landing point on the north bank. Give your answers correct to one decimal place. **4**

End of Question 12

Question 13 (15 marks) Use the Question 13 writing booklet.

(a) Find $\int_0^{2\pi} \sin^2\left(\frac{x}{2}\right)\, dx.$ **2**

(b) The diagram below shows the curve $y = \sqrt{16 - x^2}$ and the line $y = x$. **4**



The shaded region enclosed between the curve $y = \sqrt{16 - x^2}$, $y = x$ and the y-axis is rotated about the y-axis.

Find the volume of the solid generated. Give your answer correct to the nearest unit^3.

(c) Find the equation of the tangent to the curve $y = (1 + x)^2 \tan^{-1} x$ at the point $(1,0)$. **3**

(d) Find the solution of the differential equation $y' = 2e^{x-y}$ given $y(1) = \ln(1 + 2e)$. **3**

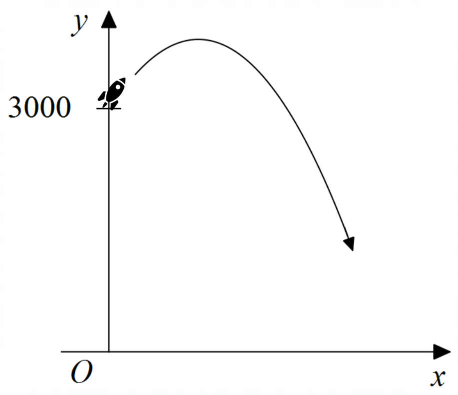
(e) A particle is moving such that, at time, *t* seconds, its displacement, *x* metres, satisfies the equation $t = 2 - \frac{1}{e^{3x}}$. Find the acceleration of the particle after 1 second. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 writing booklet.

(a) Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers n $(n \geq 1)$. **3**

(b) A missile is at an elevation of 3000 m and ascending with a velocity of 300 ms^{-1} at an angle of inclination of 60° when it stops accelerating due to a malfunction, causing its rocket to cut out.



i) Given that the only force acting on the missile is now gravity ($g = 10\ m/s^2$), show that the velocity $\underset{\sim}{v}(t) = 150\underset{\sim}{i} + (-10t + 150\sqrt{3}\,)\underset{\sim}{j}$. **3**

ii) After the rocket cuts out, how far will it travel in a horizontal direction before it hits the ground? Give your answer correct to the nearest metre. **3**

(c) At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3. The surface area of the cube is increasing at a constant rate of 8 cm^{2}s^{-1}.

i) Show that $\dfrac{dx}{dt} = \dfrac{k}{x}$ where k is a constant. **1**

ii) Show that $\dfrac{dV}{dt} = 2V^{\frac{1}{3}}$. **2**

iii) Given that $V = 8$ when $t = 0$, solve the differential equation in part (ii), and find the value of t when $V = 16\sqrt{2}$. **3**

End of Paper.

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Section I

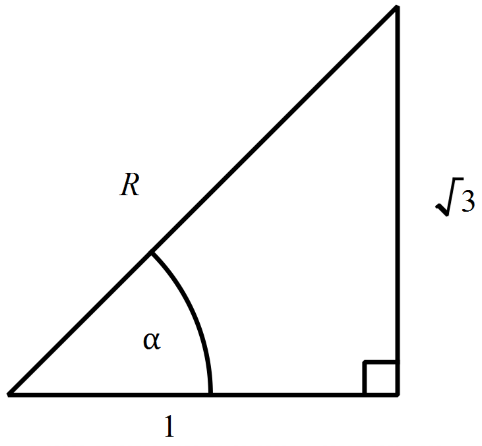
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10.

1.	A PIN is to be chosen using four of the digits 0 through 9. Digits cannot be repeated. How many possible PINs are there? A. 256 B. 210 C. 10 000 D. 5040
Answer	${}^{10}P_4 = 5040$
2.	An object's velocity is given by the vector $\binom{3}{-5}$. What is the magnitude of the object's velocity? A. -2 B. 34 C. $\sqrt{34}$ D. $\sqrt{8}$
Answer	Velocity = $\sqrt{3^2 + (-5)^2} = \sqrt{34}$
3.	The polynomial $P(x) = x^3 - 12x^2 + 21x + k$ has a double root. What are the possible values of k? A. $k = -1$ or $k = -7$ B. $k = 1$ or $k = 7$ C. $k = 10$ or $k = -98$ D. $k = -10$ or $k = 98$

Answer	For $P(x)$ to have a double root, its derivative must have a root in common with $P(x)$. $P'(x) = 3x^2 - 24x + 21 = 0$ $3(x^2 - 8x + 7) = 0$ $3(x-7)(x-1) = 0$ So the root is either $x = 7$ or $x = 1$. $P(7) = 7^3 - 12(7)^2 + 21(7) + k = 0$ so $k = 98$ $P(1) = 1^3 - 12(1)^2 + 21(1) + k = 0$ so $k = -10$
4.	What is the derivative of $\sin^{-1} 3x$? A. $-\frac{1}{\sqrt{1-9x^2}}$ B. $\frac{3}{\sqrt{1-9x^2}}$ C. $-\frac{1}{\sqrt{1-3x^2}}$ D. $-\frac{3}{\sqrt{1-9x^2}}$
Answer	Derivative of $sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$ $f(x) = 3x$ $f'^{(x)} = 3$ $\sin^{-1} 3\text{x} = \frac{3}{\sqrt{1-(3x)^2}}$

5.	Find the primitive function of $\frac{dy}{dx} = \sin x \cos^3 x$. A. $y = \frac{1}{4}\cos^4 x + C$ B. $y = -\frac{1}{4}\cos^4 x + C$ C. $y = \frac{1}{8}\sin^2 x \cos^4 x + C$ D. $y = -\frac{1}{8}\sin^2 x \cos^4 x + C$
Answer	We know $\frac{d}{dx}\cos x = -\sin x$ Thus by the reverse chain rule, $\int \sin x \cos^3 x\, dx = -\frac{1}{4}\cos^4 x + C$ Alternate solution: using substitution $u = cosx$
6.	Write the expression $y = \sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$. A. $2\sin\left(x + \frac{\pi}{4}\right)$ B. $\sin\left(x + \frac{\pi}{3}\right)$ C. $2\sin\left(x + \frac{\pi}{3}\right)$ D. $2\sin\left(x - \frac{\pi}{3}\right)$

Answer



$$R = \sqrt{4}$$
$$= 2$$

$$\tan \alpha = \sqrt{3}$$
$$\alpha = \frac{\pi}{3}$$

$$\therefore \sin x + \sqrt{3} \cos x = 2 \sin\left(x + \frac{\pi}{3}\right)$$

7. What is the value of $\sin 2x$ given that $\sin x = \frac{2\sqrt{3}}{4}$ and x is obtuse?

A. $-\frac{\sqrt{3}}{4}$

B. $-\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{4}$

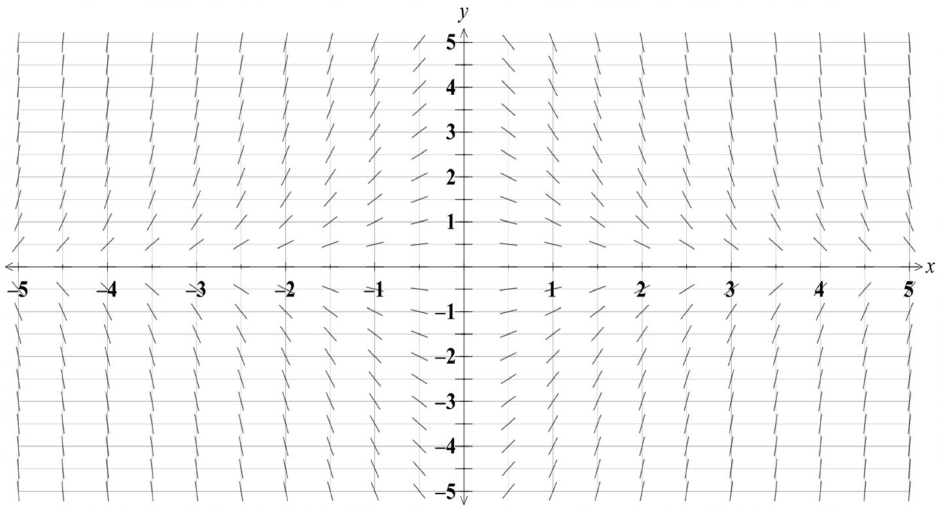
D. $\frac{\sqrt{3}}{2}$

Answer

$$4^2 = \left(2\sqrt{3}\right)^2 + a^2$$
$$a^2 = 16 - 12$$
$$a = 2$$

$$\sin 2x = 2\sin x\cos x$$
$$= 2 \times \frac{2\sqrt{3}}{4} \times -\frac{2}{4} = -\frac{\sqrt{3}}{2}$$

8.	The equation $y = e^{ax}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$. What are the possible values of a? A. $a = -2$ or $a = 3$ B. $a = -1$ or $a = 6$ C. $a = 2$ or $a = -3$ D. $a = 1$ or $a = -6$
Answer	$y = e^{ax}, \frac{dy}{dx} = ae^{ax}, \frac{d^2y}{dx^2} = a^2e^{ax}$ $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ $a^2e^{ax} + ae^{ax} - 6e^{ax} = 0$ $(a^2 + a - 6)e^{ax} = 0$ $(a-2)(a+3)e^{ax} = 0$ $\therefore a = 2$ or $a = -3$
9.	Given that $f(x) = e^x - 1$ and $y = f^{-1}(x)$, find an expression for $\frac{dy}{dx}$. A. $\frac{1}{e^x - 1}$ B. $\frac{1}{x+1}$ C. $\ln x$ D. $\ln(x+1)$

Answer	$f(x) = e^x - 1$ Let $y = e^x - 1$ For $f^{-1}(x)$ we have $x = e^y - 1$ which also gives $e^y = x + 1$ $\frac{dx}{dy} = e^y$ $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ $= \frac{1}{e^y}$ $= \frac{1}{x+1}$
10.	Which of the following differential equations could be represented by the slope field diagram below?  A. $y' = -xy$ B. $y' = xy$ C. $y' = -x^2y$ D. $y' = x^2y$
Answer	The slope in the 1st and 3rd quadrants is always negative. The slope in the 2nd and 4th quadrants is always positive. This rules out options B, C and D.

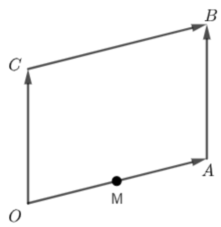
Section II

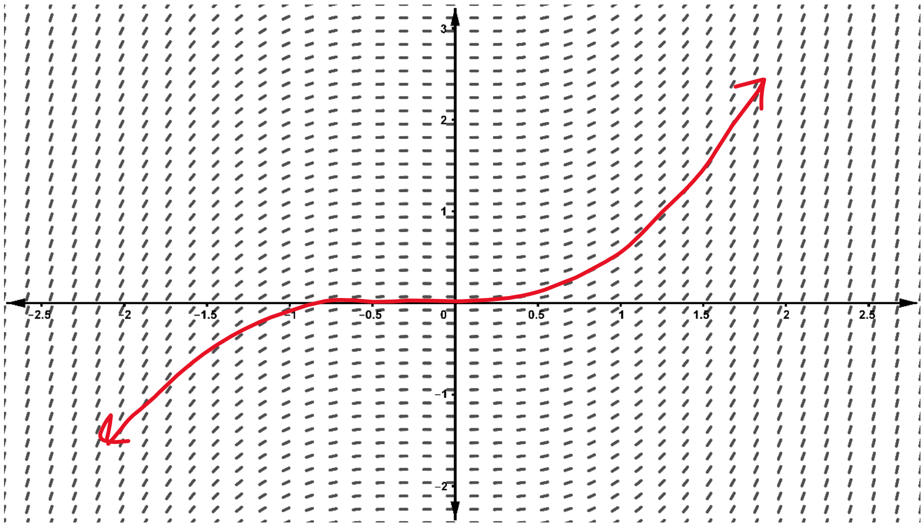
60 marks
Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

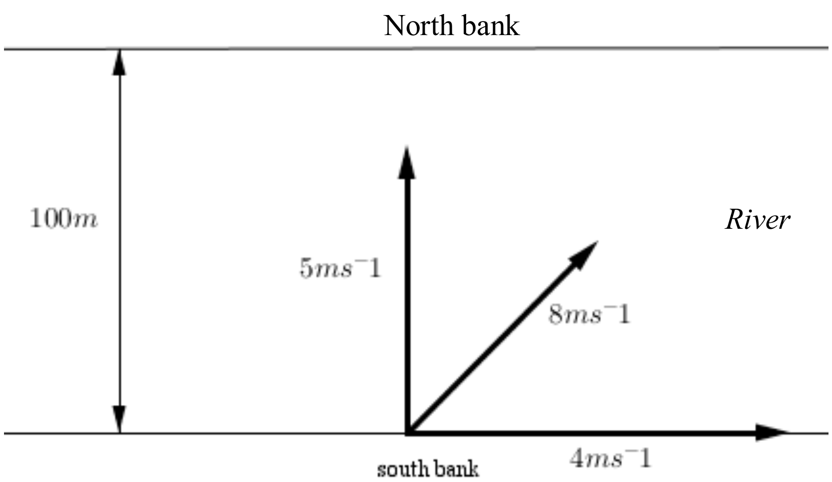
Answer each question in the appropriate writing booklet. Extra writing booklets are available.
In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

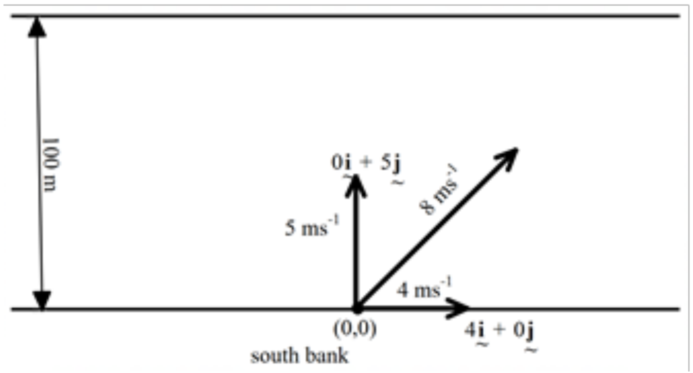
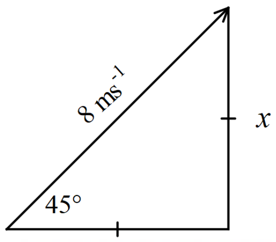
Question 11 (15 marks) Use the Question 11 writing booklet.		
(a)	Write a cartesian equation to represent the following parametric equations. $x = 2\sin\theta$ $y = 2\cos\theta$	**2**
Answer	$x^2 = 4sin^2\theta$ $y^2 = 4cos^2\theta$ $x^2 + y^2 = 4sin^2\theta + 4cos^2\theta$ $x^2 + y^2 = 4(sin^2\theta + cos^2\theta)$ $x^2 + y^2 = 4$	
(b)	Solve $\frac{1-x}{x-5} \geq 1$.	**2**
Answer	$\frac{1-x}{x-5} \geq 1$ $\frac{1-x}{x-5} \times (x-5)^2 \geq (x-5)^2$ $(1-x)(x-5) \geq (x-5)^2$ $(x-5)^2 - (1-x)(x-5) \leq 0$ $(x-5)(2x-6) \leq 0 \quad , x \neq 5$ Solution : $3 \leq x < 5$	
(c)	Consider the vectors $\underset{\sim}{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ onto $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. i) Using the dot product, show that these two vectors are not perpendicular to each other.	**1**

	$\underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{q}} = \lvert\underset{\sim}{\mathbf{p}}\rvert\lvert\underset{\sim}{\mathbf{q}}\rvert \cos\theta$ where θ is the angle between the vectors. $\lvert\underset{\sim}{\mathbf{p}}\rvert = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$ $\lvert\underset{\sim}{\mathbf{q}}\rvert = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$ $\underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{q}} = 5 \times \sqrt{5}\cos\theta$ $\underset{\sim}{\mathbf{p}} \cdot \underset{\sim}{\mathbf{q}} = -4 - 6$ $= -10$ $\therefore \cos\theta = -\frac{10}{5\sqrt{5}}$ Since $\cos\theta \neq 0, \theta \neq 0$, hence these two vectors are not perpendicular to each other	
(d)	ii) Find the vector projection of $\underset{\sim}{p}$ onto $\underset{\sim}{q}$.	**2**
	$\text{Proj}_{\underset{\sim}{q}}(\underset{\sim}{p}) = \frac{\underset{\sim}{p}.\underset{\sim}{q}}{\lvert\underset{\sim}{q}\rvert^2} \cdot \underset{\sim}{q}$ $\underset{\sim}{p} = \binom{4}{-3}$ onto $\underset{\sim}{q} = \binom{-1}{2}$. $\underset{\sim}{p}.\underset{\sim}{q} = 4 \times -1 + -3 \times 2 = -10$ $\lvert\underset{\sim}{q}\rvert^2 = (-1)^2 + 2^2 = 5$ $\text{Proj}_{\underset{\sim}{q}}(\underset{\sim}{p}) = \frac{-10}{5} \cdot \binom{-1}{2}$ $= -2\binom{-1}{2}$ $= \binom{2}{-4}$ $= 2\underset{\sim}{i} - 4\underset{\sim}{j}$ Or Alternate solutions $\hat{\underset{\sim}{q}} = -\frac{i}{\sqrt{5}} + \frac{2j}{\sqrt{5}}$ $\text{Proj}_{\underset{\sim}{q}}\underset{\sim}{p} = (\lvert\underset{\sim}{p}\rvert\cos\theta)\hat{\underset{\sim}{q}}$ $= \left(5 \times \frac{-10}{5\sqrt{5}}\right) \times \left(-\frac{j}{\sqrt{5}} + \frac{2j}{\sqrt{5}}\right)$ $= +\frac{2\sqrt{5}i}{\sqrt{5}} - \frac{2 \times 2\sqrt{5}j}{\sqrt{5}}$ $\therefore \text{Proj}_{\underset{\sim}{q}}\underset{\sim}{p} = 2i - 4j$	
(e)	In the diagram, $OABC$ is a parallelogram. The vector $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. M is the midpoint of OA. Write an expression in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ to represent the vector $\overrightarrow{MB}$. 	**1**
Answer	$\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM} = (\underset{\sim}{a} + \underset{\sim}{c}) - \frac{1}{2}\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c}$ Or $\overrightarrow{MA} + \overrightarrow{AB} = \overrightarrow{MB}$ $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{c}$ $\overrightarrow{OA} = \underset{\sim}{a}$ $\overrightarrow{MA} = \frac{\underset{\sim}{a}}{2}$ $\therefore \overrightarrow{MB} = \frac{\underset{\sim}{a}}{2} + \underset{\sim}{c}$	

(f)	Sketch a possible solution curve passing through the origin for the following direction field. 	**1**
(g)	Find $\int_0^{\ln 2} \frac{e^{2x}}{1+e^{4x}}\,dx$ by using the substitution $u = e^{2x}$. Give your answer correct to 4 decimal places.	**3**
Answer	$e^{4x} = (e^{2x})^2$. $u = e^{2x}, \frac{du}{dx} = 2e^{2x} \rightarrow \frac{1}{2}du = e^{2x}dx$ Changing the limits of the integral: When $x = 0, u = e^{2(0)} = 1$ When $x = \ln 2, u = e^{2\ln 2} = e^{\ln 4} = 4$ The integral becomes: $\int_1^4 \frac{1}{1+u^2} \times \frac{1}{2}du = \frac{1}{2}[\tan^{-1} u]_1^4 = \frac{1}{2}(\tan^{-1} 4 - \tan^{-1} 1] = \frac{1}{2}(\tan^{-1} 4 - \tan^{-1} 1) \approx$ 0.2702	
(h)	Use the substitution $t = \tan\frac{x}{2}$ to find in radians, correct to 2 decimal places, the solutions of the equation $\cos x - 3\sin x + 3 = 0$ for $0 \leq x \leq 2\pi$.	**3**
Answer	$\frac{1-t^2}{1+t^2} - 3\left(\frac{2t}{1+t^2}\right) + 3 = 0$ $1 - t^2 - 6t + 3 + 3t^2 = 0$ $t^2 - 3t + 2 = 0$ $(t-2)(t-1) = 0$ $\therefore$ *for* $x \neq \pi,\ 0 \leq x \leq 2\pi$ $\tan\frac{x}{2} = 2$ *or* $\tan\frac{x}{2} = 1$ $\frac{x}{2} = \tan^{-1} 2$ $\frac{x}{2} = \frac{\pi}{4}$ $x = 2\tan^{-1} 2$ $x = \frac{\pi}{2}$ $\therefore x \approx 2 \cdot 21,\ 1 \cdot 57$ *for* $x = \pi$, $\cos\pi - 3\sin\pi + 3 = 2$ $\therefore x \neq \pi$	

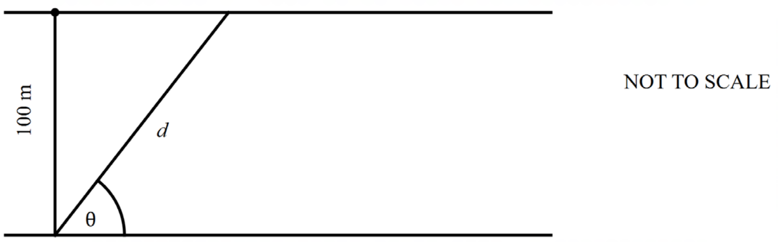
Question 12 (15 marks) Use the Question 12 writing booklet.		
(a)	i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.	**1**
Answer	$\sin(A+B)+\sin(A-B)=\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B$ $=2\sin A\cos B$	
	ii) Hence, find $\displaystyle\int_0^{\frac{\pi}{2}} \sin 4x\cos 2x\,dx$	**3**
Answer	$sin4x\,cos\,2x=\frac{1}{2}(sin(4x+2x)+sin(4x-2x))$ $\therefore \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}(sin(4x+2x)+sin(4x-2x))dx$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}(sin6x+sin2x)dx$ $\frac{1}{2}\left[-\frac{1}{6}cos6x\right]_0^{\pi/2}+\frac{1}{2}\left[-\frac{1}{2}cos2x\right]_0^{\pi/2}$ $=-\frac{1}{12}\left[cos6\times\frac{\pi}{2}-cos0\right]-\frac{1}{4}\left[cos2\times\frac{\pi}{2}-cos0\right]$ $=\frac{1}{12}[-1-1]-\frac{1}{4}[-1-1]$ $=\frac{2}{3}$	
(b)	Find $\displaystyle\int_0^{\frac{5}{2}} \frac{1}{25+4x^2}\,dx$. Leave your answer in exact form.	**2**

Answer	$\int_0^{\frac{5}{2}} \frac{1}{25+4x^2}\ dx$ $\frac{1}{2}\int_0^{\frac{5}{2}} \frac{1\times 2}{5^2+(2x)^2} dx$ $\frac{1}{2}\times\frac{1}{5}\left[tan^{-1}\frac{2x}{5}\right]_0^{5/2}$ $\frac{1}{10}\left[tan^{-1}\frac{2}{5}\times\frac{5}{2} - tan^{-1}\frac{2}{5}\times 0\right]$ $=\frac{1}{10}\times\frac{\pi}{4}$ $=\frac{\pi}{40}$	
(c)	The polynomial equation $x^3 - 5x^2 + x + 3 = 0$ has roots α, β and γ. Find $\alpha^2 + \beta^2 + \gamma^2$.	**3**
Answer	$\alpha + \beta + \gamma = 5$ $\alpha\beta + \beta\gamma + \alpha\gamma = 1$ $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $= 5^2 - 2\times 1$ 23	
(d)	Sienna intends to row her boat from the south bank of a river to meet with her friends on the north bank, as shown in the diagram below. The river is 100 metres wide. Sienna's rowing speed is 5 ms^{-1} when the water is still. The river is flowing east at a rate of 4 ms^{-1}. Sienna's boat is also being impacted by a wind blowing from the south-west, which pushes the boat at 8 ms^{-1}. She starts rowing across the river by steering the boat such that it is perpendicular to the south bank.  	

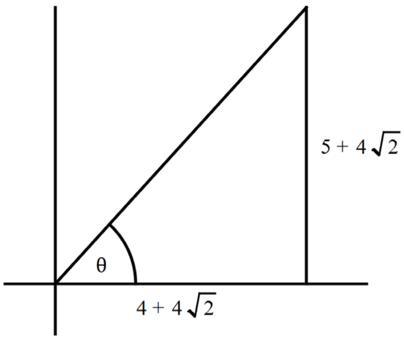
	i) Show that the velocity of Sienna's boat can be expressed as the component vector: $(4 + 4\sqrt{2})\underset{\sim}{\mathbf{i}} + (5 + 4\sqrt{2})\underset{\sim}{\mathbf{j}}$	**2**
Answer	 Component vectors of the wind 	
	$8^2 = 2x^2$ $x^2 = 32$ $x = \sqrt{32}$ $= 4\sqrt{2}$ Wind vector is $4\sqrt{2}\,\underset{\sim}{\mathbf{i}} + 4\sqrt{2}\,\underset{\sim}{\mathbf{j}}$ Rowing vector is $0\,\underset{\sim}{\mathbf{i}} + 5\,\underset{\sim}{\mathbf{j}}$ Water flow vector is $4\,\underset{\sim}{\mathbf{i}} + 0\,\underset{\sim}{\mathbf{j}}$ Resultant velocity $= 4\underset{\sim}{\mathbf{i}} + 0\underset{\sim}{\mathbf{j}} + 0\underset{\sim}{\mathbf{i}} + 5\underset{\sim}{\mathbf{j}} + 4\sqrt{2}\,\underset{\sim}{\mathbf{i}} + 4\sqrt{2}\,\underset{\sim}{\mathbf{j}}$ $= (4 + 4\sqrt{2})\underset{\sim}{\mathbf{i}} + (5 + 4\sqrt{2})\underset{\sim}{\mathbf{j}}$	

	ii) Determine the distance rowed (in metres) and time taken (in seconds) by Sienna to travel from the starting point to her landing point on the north bank. Give your answers correct to one decimal place.	**4**
Answer	The velocity is equal to the magnitude of the resultant velocity vector from (a).	

$$v = \sqrt{(4 + 4\sqrt{2})^2 + (5 + 4\sqrt{2})^2}$$
$$= 14.38135517$$
$$\approx 14.38 \text{ ms}^{-1}$$



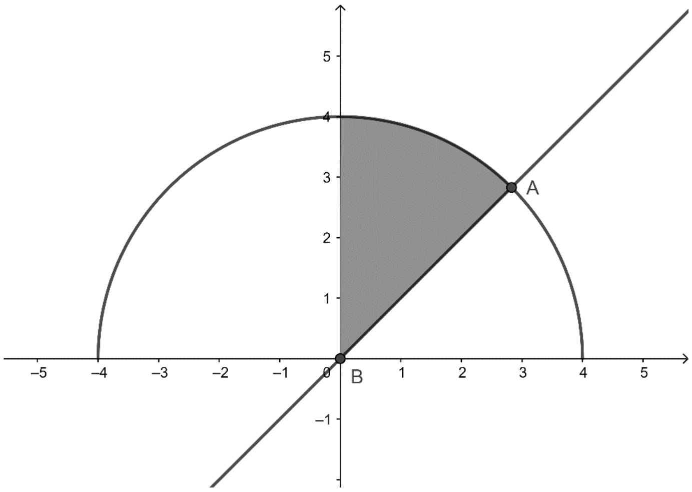
$$\tan\theta = \frac{5 + 4\sqrt{2}}{4 + 4\sqrt{2}}$$
$$\theta = \tan^{-1}\left(\frac{5 + 4\sqrt{2}}{4 + 4\sqrt{2}}\right)$$
$$= 47.81827238°$$
$$\approx 48°$$



The distance from start to finish is given by *d*.

$$\sin\theta = \frac{100}{d}$$
$$d = \frac{100}{\sin 48°}$$
$$= 134.9493465 \text{ m}$$
$$\approx 135 \text{ m}$$

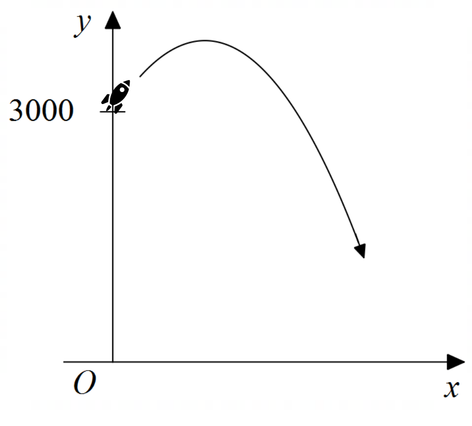
$$\text{Time taken} = 134.9493465 \text{ m} \div 14.38 \text{ ms}^{-1}$$
$$= 9.384516448$$
$$\approx 9.4 \text{ seconds}$$

Question 13 (15 marks) Use the Question 13 writing booklet.		
(a)	Find $\int_0^{2\pi} \sin^2\left(\frac{x}{2}\right)\, dx$.	**2**
Answer	$\int_0^{2\pi} \sin^2\frac{x}{2} dx$ $= \int_0^{2\pi} \frac{1}{2}(1-\cos x)dx$ $= \frac{1}{2}[x - \sin x]_0^{2\pi}$ $= \frac{1}{2}[2\pi - 0) - (0-0)]$ $= \pi$	
(b)	The diagram below shows the curve $y = \sqrt{16 - x^2}$ and the line $y = x$.  The shaded region enclosed between the curve $y = \sqrt{16-x^2}$, $y = x$ and the y-axis is rotated about the y-axis.	**4**
	Find the volume of the solid generated. Give your answer correct to the nearest unit^3.	

Answer	Find the point of intersection of both the graphs first. $x = \sqrt{16 - x^2}$ $x^2 = 16 - x^2$ $2x^2 = 16$ $x = \pm 2\sqrt{2}$ Discard the negative value because the point is on the positive side of *x-axis*. $y = x = 2\sqrt{2}$ $x = \sqrt{16 - y^2}$ $V = \pi\left(\int_0^{2\sqrt{2}} y^2 dx + \int_{2\sqrt{2}}^{4}(16 - y^2)dx\right)$ $V = \pi\left(\left[\frac{y^3}{3}\right]_0^{2\sqrt{2}} + \left[16y - \frac{y^3}{3}\right]_{2\sqrt{2}}^{4}\right)$ $V = \pi\left[\frac{(2\sqrt{2})^3}{3} + 16 \times 4 - \frac{4^3}{3} - 16 \times 2\sqrt{2} + \frac{(2\sqrt{2})^3}{3}\right]$ $V \approx 39\ unit^3$	
(c)	Find the equation of the tangent to the curve $y = (1 + x)^2 \tan^{-1} x$ at the point $(1,0)$.	**3**
Answer	First find the gradient of the tangent by differentiating the function: $y' = (1 + x)^2 \times \frac{1}{1 + x^2} + 2(1 + x)\tan^{-1} x$ $y'(1) = 2 + 4\tan^{-1} 1$ $y'(1) = 2 + 4 \times \frac{\pi}{4}$ $y' = 2 + \pi$ Equation of the tangent $y - 0 = (2 + \pi)(x - 1)$ $y = (2 + \pi)x - 1 - \pi$	

(d)	Find the solution of the differential equation $y' = 2e^{x-y}$ given $y(1) = \ln(1+2e)$.	**3**
Answer	$y' = 2e^{x-y}$ $\frac{dy}{dx} = \frac{2e^x}{e^y}$ $e^y dy = 2e^x dx$ Integrate both sides; $\int e^y \, dy = \int 2e^x dx$ $e^y = 2e^x + c$ $sub\ x = 1\ and\ y = \ln(1+2e)$ $e^{\ln(1+2e)} = 2e + c$ $1 + 2e = 2e + c$ $c = 1$ $\therefore e^y = 2e^x + 1$ $y = \ln(2e^x + 1)$	
(e)	A particle is moving such that, at time, t seconds, its displacement, x metres, satisfies the equation $t = 2 - \frac{1}{e^{3x}}$. Find the acceleration of the particle after 1 second.	**3**

Answer		
	$t = 2 - \frac{1}{e^{3x}}$ $= 2 - e^{-3x}$ $\frac{dt}{dx} = 3e^{-3x}$ $= \frac{3}{e^{3x}}$ $\therefore \frac{dx}{dt} = \frac{e^{3x}}{3}$ ie $v = \frac{e^{3x}}{3}$ $x = \frac{1}{3}\ln(\frac{1}{2-t})$ $v = \frac{e^{3\times\frac{1}{3}\ln\left(\frac{1}{2-t}\right)}}{3}$ $v = \frac{1}{3(2-t)}$ $\frac{dx}{dt} = \frac{1}{3(2-t)}$ $a = \frac{d^2x}{dt^2} = \frac{1}{3}\left(\frac{1}{2-t}\right)$ when $t = 1$ $a = \frac{1}{3} \times \frac{1}{2-1}$ $= \frac{1}{3}\text{ ms}^{-2}$	

Question 14 (15 marks) Use the Question 14 writing booklet.		
(a)	Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers n $(n \geq 1)$.	**3**
Answer	Step 1: To prove true for $n = 1$ $$4^1 + 14 = 18$$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $$4^k + 14 = 6\text{m}$$ where m is an integer Step 3: To prove true for $n = k + 1$ $$4^{k+1} + 14 = 6\text{p}$$ where p is an integer $$\begin{aligned} \text{LHS} &= 4^{k+1} + 14 \\ &= 4(4^k) + 14 \\ &= 4(4^k) + 4 \times 14 - 3 \times 14 \\ &= 4(4^k + 14) - 3 \times 14 \\ &= 4(6\text{m}) - 42 \\ &= 6(4\text{m} - 7) = 6p \\ &= \text{RHS} \end{aligned}$$ Step 4: True by induction	
(b)	A missile is at an elevation of 3000 m and ascending with a velocity of 300 ms^{-1} at an angle of inclination of 60° when it stops accelerating due to a malfunction, causing its rocket to cut out.  	
	i) Given that the only force acting on the missile is now gravity $(g = 10\ m/s^2)$, show that the velocity $\underset{\sim}{v}(t) = 150\underset{\sim}{i} + (-10t + 150\sqrt{3})\underset{\sim}{j}$.	**3**

Answer	The missile has stopped accelerating, and so the only force acting on the rocket is gravity. $\ddot{x} = 0, \ddot{y} = -10$ Integrating: $\dot{x}(t) = \int \ddot{x} dt = \int 0 dt$ $\dot{x}(t) = C_1,$ At $t = 0, \dot{x}(0) = 300 \cos 60°$ $\therefore \ C_1 = 300 \cos 60°$ Hence, $\dot{x}(t) = 300 \cos 60°$ $\dot{x}(t) = 150$ $\dot{y} = \int \ddot{y} dt = \int -10 \, dt$ $\dot{y} = -10t + C_2$ At $t = 0, \dot{y}(0) = 150\sqrt{3}$ $150\sqrt{3} = -10 \times 0 + C_2$ $\therefore \ C_2 = 150\sqrt{3}$ $\dot{y} = -10t + 150\sqrt{3}$ $\therefore \ \underset{\sim}{v}(t) = 150\underset{\sim}{i} + \left(-10t + 150\sqrt{3}\right)\underset{\sim}{j}$	

	ii) After the rocket cuts out, how far will it travel in a horizontal direction before it hits the ground? Give your answer correct to the nearest metre.	**3**
Answer	Integrate $\dot{y}$ to get an equation for y. $\dot{y} = -10t + 150\sqrt{3}$ $y = \int -10t + 150\sqrt{3}\,dt$ $y = -5t^2 + 150\sqrt{3}t + c$ Initially the particle is at $y = 3000m$. $3000 = -5(0)^2 + 150\sqrt{3}(0) + c$ $c = 3000$ $y = -5t^2 + 150\sqrt{3}t + 3000$ $\dot{x}(t) = 150$ $x = \int \dot{x}dt$ $x = 150t + c$ At $t = 0, x = 0$ Hence, $x = 150t$ Now we want to know the time when the vertical position is 0. Sub $y = 0$ $y = -5t^2 + 150\sqrt{3}t + 3000$ $-5t^2 + 150\sqrt{3}t + 3000 = 0$ $t = 15\sqrt{3} \pm 5\sqrt{51}$, discard $t = 15\sqrt{3} - 5\sqrt{51}$ and use $t = 15\sqrt{3} + 5\sqrt{51}$ Sub this value of t in $x = 150t$ $x = 150 \times (15\sqrt{3} + 5\sqrt{51})$ $= 9253.185638m$ $x \approx 9253\ m$ (to the nearest m)	

(c)	At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3. The surface area of the cube is increasing at a constant rate of 8 cm^2s^{-1}.	
	i) Show that $\frac{dx}{dt} = \frac{k}{x}$ where k is a constant.	**1**
Answer	Surface area of a cube with a side length of x is $6x^2$ $S = 6x^2$ $\frac{dS}{dx} = 12x$ Given $\frac{dS}{dt} = 8$ $\frac{dx}{dt} = \frac{dS}{dt} \div \frac{dS}{dx} = \frac{8}{12x}$ $= \frac{\frac{2}{3}}{x}$ with $k = \frac{2}{3}$	
	ii) Show that $\frac{dV}{dt} = 2V^{\frac{1}{3}}$.	**2**
Answer	$V = x^3$ (volume of a cube with side length x) $\frac{dV}{dx} = 3x^2$ $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ $= 3x^2 \times \frac{2}{3x} = 2x$ sub $x = V^{\frac{1}{3}}$ $\frac{dV}{dt} = 2V^{\frac{1}{3}}$	
	iii) Given that $V = 8$ when $t = 0$, solve the differential equation in part (ii), and find the value of t when $V = 16\sqrt{2}$.	**3**

Answer	$$\frac{dV}{dt} = 2V^{\frac{1}{3}}$$ $$V^{-\frac{1}{3}}dV = 2\mathrm{dt}$$ $$\int V^{-\frac{1}{3}}\,dV = \int 2\,\mathrm{dt}$$ $$\frac{3}{2}V^{\frac{2}{3}} = 2t + C$$ Given that $V = 8$ when $t = 0$ $$\frac{3}{2}8^{\frac{2}{3}} = 2 \times 0 + C \text{ or } C = 6$$ $$\frac{3}{2}V^{\frac{2}{3}} = 2t + 6$$ Find t when $V = 16\sqrt{2}$ $$\frac{3}{2}\left(16\sqrt{2}\right)^{\frac{2}{3}} = 2t + 6$$ $$12 = 2t + 6$$ $$t = 3 \text{ seconds}$$	

End of Paper